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JULY, 1860.

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J. D. RUNKLE, A.M., A.A.S.

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Benjamin Peirce

THE

MATHEMATICAL MONTHLY.

Vol. II. . . . JULY, 1860. . . . No. X.

PRIZE PROBLEMS FOR STUDENTS.

I. Solve the equations

$$x^2 y^2 (x^4 - y^4) = 2340,$$
$$xy (xy^3 - 1)(x^2 + y^2) = 1794,$$

by quadratics. — Communicated by E. A. HOPKINS, Cleveland, Ohio.

II. Given the lengths of the three perpendiculars dropped from any point in the plane of an equilateral triangle upon the sides; to find the segments of the sides. — Communicated by F. E. TOWER, Amherst College.

III. If e denote the edge of any regular dodecahedron, and $\beta = 36°$, prove that

$$\text{solidity} = \frac{5\,e^3 \cot^2 \beta}{2\sqrt{(4 \sin^2 \beta - 1)}} = \frac{5\,e^3 \tan^2 45}{4\sqrt{(\sin 6° \cos 24°)}}.$$

Also obtain similar formulas for the solidity of the icosahedron. — Communicated by Prof. D. W. HOYT.

IV. A given cylindrical vessel, filled with water, is placed with its base upon a horizontal plane. It is required to determine the angle of inclination to which the plane must be raised before the vessel will fall, the water being at liberty to overflow its top. The base is supposed to be fixed so as to prevent it from sliding, but not from tilting when the plane is inclined. — Communicated by Professor KIRKWOOD.

V. Bisect the attraction which a sphere of varying density exerts

upon an exterior point; that is, divide the sphere so that the two parts shall exert the same attractive force in the same direction.

Solutions of these problems must be received by September 1, 1860.

REPORT OF THE JUDGES UPON THE SOLUTIONS OF THE PRIZE PROBLEMS IN No. VII., Vol. II.

THE first Prize is awarded to JOHN A. WINEBRENER, Princeton College, N. J.

The second Prize is awarded to GEORGE C. ROUND, Wesleyan University, Middletown, Ct.

The third Prize is awarded to LEWIS FOOTE, O. C. Seminary, Cazenovia, N. Y.

PRIZE SOLUTION OF PROBLEM I.

By Miss HARRIET S. HAZELTINE, Worcester, Mass.

Prove that an arithmetic mean is greater than a geometric.

Let $x - y$ and $x + y$ denote the extremes; then x is the arithmetic and $\sqrt{(x^2 - y^2)}$ the geometric mean, and it is evident that $x = \sqrt{x^2} > \sqrt{(x^2 - y^2)}$.

SECOND SOLUTION.—Let $a > b$; then $a - b > 0$; $a^2 - 2ab + b^2 > 0$; $a^2 + 2ab + b^2 > 4ab$; $a + b > 2\sqrt{ab}$; $\therefore \frac{1}{2}(a + b) > \sqrt{ab}$. — G. S. MORISON, Harvard College.

PRIZE SOLUTION OF PROBLEM II.

Let three bodies with velocities V, V', V'', move uniformly in the same direction, in the circumference of a circle. Required the time of their conjunction, supposing them to quit a given point at the same time.

Let C denote the circumference of the circle; then, since $V - V'$ and $V - V''$ are respectively the gains of V upon V' and V'' in the same unit of time, $\frac{C}{V - V'}$ and $\frac{C}{V - V''}$ will denote the times which will elapse between the instant of starting and the conjunctions of V, V' and V, V'' respectively. And the least common multiple of

these times, or their product if they are prime to each other, will give the time which will elapse between successive conjunctions of the three bodies.

This is essentially the solution given by several of the competitors.

Prize Solution of Problem III.

By Stockwell Bettes, Boston, Mass.

The diameter of a circle inscribed in the quadrant of a second circle is equal to the side of the regular octagon circumscribed about the second circle.

Bisect the quadrant by the line AK, and draw the tangent CB at K. Next, bisect the angles CAK and KAB, and IL will be the side of the circumscribed octagon. The centre of the circle inscribed in the triangle CAB is the same as that inscribed in the quadrant, and it is at the intersection, H, of the lines bisecting the angles of the triangle. The triangles AKL and BKH are equal, and therefore $KH = KL$. But $KI = KL$; therefore $IL = 2\,HK$.

Second Solution. — LK and LE, tangents at K and E, are half sides of the octagon. Draw LN parallel to AB; then angle $KHL = KAB = KLH$, $\therefore KH = KL = LE = HM = HN$, therefore H is the centre of the inscribed circle. — John R. Emery, Princeton College, New Jersey.

Prize Solution of Problem IV.

By Cadet Arthur H. Dutton, West Point, N. Y., and Hiram L. Gear, Marietta College, Ohio.

Required the locus of the centres of the circles inscribed within all the right-angled triangles which can be inscribed in a given semicircle.

Let P be the centre of a circle inscribed in any right-angled triangle, ABC, which can be inscribed in the semicircle AEB. Draw the diameter EF perpendicular to AC, and join BF, PC,

and $F\,C$. Because $B\,F$ bisects the angle $A\,B\,C$, it passes through the point P. But $F\,P\,C = P\,B\,C + B\,C\,P = A\,C\,F + P\,C\,A = P\,C\,F$.

Therefore the triangle $F\,P\,C$ is isosceles, and the point P is in the circumference of a circle of which F is the centre, and radius $F\,C = O\,C\sqrt{2}$.

Second Solution. — The centre of the inscribed circle is at the intersection of the lines bisecting the angles at the base, and as the sum of these angles is constant, the half sum is also constant; and hence the vertical angle of this second triangle is constant, and since the base is constant, the locus of these points must be in a circle. — John A. Winebrener, Princeton College, N. Y.

Third Solution. — Let $(x\,y)$ be the co-ordinates of the point P, O being the origin, and r the radius of circle $A\,B\,C$. But $\tan(P\,A\,C + P\,C\,A) = \tan 45° = 1$, and since $\tan P\,A\,C = \frac{y}{r+x}$, $\tan P\,C\,A = \frac{y}{r-x}$, we have by the usual formula, $x^2 + y^2 + 2\,r\,y = r^2$; or changing the origin to F, $x^2 + y^2 = 2\,r^2$, therefore the locus of the point P is a circle whose centre is F.

All the analytical solutions are essentially the same.

Prize Solution of Problem V.

By W. F. Osborne, Wesleyan University, Middletown, Ct.

From a box containing a very large number of white and black balls, of each an equal number, three balls are taken at random and placed in a bag without being seen. A takes a ball at random from the bag, observes its color, and replaces it four times in succession. The ball was white on each of the four drawings. What are the respective probabilities that the bag contains 1, 2, or 3 white balls?

The balls as drawn from the box and placed in the bag will be, either no white, $b\,b\,b$; one white, $w\,b\,b$, $b\,w\,b$, $b\,b\,w$; two white, $w\,w\,b$,

w b w, *b w w*; or three white, *w w w*. Hence, before drawing from the bag, the chances that the balls are one white, two white, three white are respectively $\frac{3}{8}$, $\frac{3}{8}$, $\frac{1}{8}$.

Now, if there is only one white ball in the bag, the chances that it will be drawn four successive times are $(\frac{1}{3})^4$; if two of the balls are white, the chances are $(\frac{2}{3})^4$; if three of the balls are white the chances are $(\frac{3}{3})^4$. Combining, we get $\frac{3}{8}(\frac{1}{3})^4 = \frac{1}{216}$, $\frac{3}{8}(\frac{2}{3})^4 = \frac{16}{216}$, $\frac{1}{8}(\frac{3}{3})^4 = \frac{27}{216}$, as the *a priori* probabilities that a white ball was drawn from the bag four times in succession. But since a white ball is drawn four successive times, the respective probabilities are

$$\frac{1}{1+16+27} = \frac{1}{44},\ \frac{16}{1+16+27} = \frac{16}{44},\ \frac{27}{1+16+27} = \frac{27}{44}.$$

NOTES AND QUERIES.

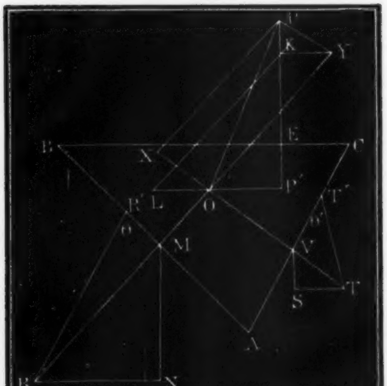
1. *Notes on the Inclined Plane and the Wedge.* — Let $A\,B\,C$ represent any inclined plane; W the weight, placed at O; P the power, which is constant and may be denoted by $O\,a$, the radius of a circle; and R the reaction in the line $O\,D$ perpendicular to the plane. The condition of equilibrium will be represented by the three sides of a triangle. R must always act in the line $O\,D$, W must be perpendicular to the horizon, and P may vary in direction, and will give different values for R and W in different positions.

When P acts in the line $O\,a$,

W must be in the same line, and therefore $P = W$ and $R = 0$. When P acts in the line $O g$, R must act in the same line, and therefore $P = R$ and $W = 0$. All the positions including the limits are then represented as follows: —

$P = O\,a$	$W = O\,a$	and $R = 0$ (min.)
$P = O\,b$	$W = b\,b'$	" $R = O\,b'$
$P = O\,c$	$W = c\,c'$ (max.)	" $R = O\,c'$
$P = O\,d$	$W = d\,d'$	" $R = O\,d'$ (max.)
$P = O\,e$	$W = e\,e'$	" $R = O\,e'$
$P = O\,g$	$W = 0$ (min.)	" $R = O\,g$.

In general W increases from an equality with P to its maximum value when P acts parallel to the plane, and afterwards decreases again to its minimum value, or zero, when P acts perpendicular to the plane. R increases from its minimum value, or zero, to its maximum, when P acts parallel to the base of the plane, then decreases again to equality with P, when P acts perpendicularly to the plane. When P acts in directions beyond the limits $O\,a$ and $O\,g$, negative results for W and R are obtained, of an interesting character.



Let $A\,B\,C$ represent any wedge whatever; also $R\,R'$ and $T\,T'$ any resistances whatever, acting at any angles, θ and θ'. Resolve $R\,R'$ into $R\,M$ and $R'\,M$; and $R\,M$ into $R\,N$ and $M\,N$. Also in like manner resolve $T\,T'$ into $T\,V$ and $T'\,V$; and $T\,V$ into $V\,S$ and $S\,T$. Produce $R\,M$ and $T\,V$ until they meet in O, and beyond O take $O\,X = V\,T$, and $O\,Y = R\,M$. Next, complete the parallelogram $O\,X\,P\,Y$; then draw the diagonal $O\,P$, and lastly, resolve $O\,P$ into $O\,P'$ and $P\,P'$. $M\,N$ represents the effective force

of $R R'$, and $V S$ that of $T T'$. Therefore when there is an equilibrium, $M N + V S = P P'$.

From Y draw $Y K$ perpendicular to $P P'$, and complete the parallelogram $Y K L O$. The triangles $R M N$ and $L K P'$ are equal, since the sides are respectively parallel, and $L K = O Y = R M$ by construction; therefore $P' K = M N$. The triangles $P K Y$ and $T V S$ are equal, for a similar reason, and $P K = V S$. Consequently $M N + V S = P' K + K P = P P'$. This method of resolution also gives the place E, where the power $P P'$ must be applied to produce equilibrium. When the prolongation of $R M$ and $T V$ meet in a point O beyond $B C$, either to the right or left, negative results are obtained. — Prof. John L. Campbell, Wabash College, Crawfordsville, Ind.

2. *Law of Gravity.* Solution of the problem on page 204, Vol. II. — Two similar systems of bodies have all corresponding linear dimensions in the same ratio to each other; that is, the linear dimensions and the distances apart of the bodies are in the one system in a fixed ratio to those of the other.

In order that two similar systems may remain similar for successive instants of time, it is necessary that the motions of corresponding bodies be in the same relative directions *inter se*, and that the velocities of corresponding motions be proportional to the linear dimensions of the two systems.

If, now, these motions be continually modified by the action of central fixed forces, or by forces dependent directly upon the masses of the bodies, and upon some function of their distances apart, then, since the changes in the motions must also be proportional to the dimensions of the systems and to the motions themselves, the values of these central forces will be proportional to the same dimensions. But the forces, so far as they are dependent upon the masses, are proportional to the cubes of the linear dimensions; hence, so far as

they depend upon the distances, they must be inversely proportional to the squares of these dimensions, in order that on the whole, from the masses and distances combined, they may be simply dependent on the first power of the distances, dimensions, and velocities of the bodies. Hence the law of gravity, and conversely corresponding motions of revolution or oscillation in two similar systems must, by the law of gravity, have the same periods, since the dimensions of the paths or orbits of these motions, their velocities, and their changes of velocity, are all proportional to the dimensions of the systems.

Hence all measures of time, whether by periods of orbital, of rotary, or of oscillatory motions, are by the law of gravity independent of the dimensions of the material universe; and if the solar sytem had been constructed on the scale of a common planetarium, it would still have moved, by virtue of the forces inherent in matter, *pari passu*, through the same phases of motion and configuration, with the same periods as now. — W.

3. Develop the Naperian logarithm of x into a series. —
Put $x = y + 1$; then $dx = dy$, and $\frac{dx}{x} = \frac{dy}{y+1}$. But by division

$$\frac{1}{y+1} = 1 - y + y^2 - y^3 + y^4 \text{ \&c.}$$

$$\therefore \frac{dx}{x} = \frac{dy}{y+1} = dy - y\,dy + y^2\,dy - y^3\,dy + y^4\,dy \text{ \&c.}$$

By integration

$$\log x = y - \tfrac{1}{2}y^2 + \tfrac{1}{3}y^3 - \tfrac{1}{4}y^4 + \tfrac{1}{5}y^5 \text{ \&c.}$$

Restoring the value of $y = x - 1$, we get

$$\log x = (x-1) - \tfrac{1}{2}(x-1)^2 + \tfrac{1}{3}(x-1)^3 - \tfrac{1}{4}(x-1)^4 \text{ \&c.}$$

— Artemas Martin, Franklin, Pa.

4. *Note on Right-angled Triangles.* — I have for many years kept on hand for my own convenience a list of the fifty two right-angled triangles described by Professor Hoyt in the May number of the Monthly. I prepared the series by using the formula

$$(a^2 + b^2)^2 = (a^2 - b^2)^2 + (2\,a\,b)^2,$$

which is easily seen to be true, and in which any numbers whatever may take the places of a and b. If for these letters we substitute the natural numbers in succession we shall obtain thirty sets of numbers no one of which will exceed a hundred; *fourteen* of these however are equimultiples of some of the others, and *twenty-two* other multiple sets may be found within the same limit. We thus find, as Professor HOYT has done, *fifty-two* right-angled triangles whose sides are expressed by integral numbers not exceeding one hundred, and *sixteen* of which are dissimilar in form. I cannot now call to mind where I found the formula given above. — Prof. E. S. SNELL, Amherst College, Mass.

5. *Note on Equal Temperaments.* — It is assumed that the number of vibrations in a given time, producing a musical tone, is to the number producing its octave as 1 is to 2; that the numbers in like manner corresponding to a note and its "fifth" are to each other as 1 to 1.5; and that in a scale of equal temperament the numbers corresponding to the successive tones are in geometrical progression. Required, the number of equal intervals into which an octave must be divided, so as to have one of the tones approximate nearly to a "fifth." Let $z =$ the ratio in the geometrical progression, $y =$ the number of intervals approximating the fifth. $x =$ the number of intervals in the octave. Then

(1) $z^y = 1.5$ nearly, (2) $z^x = 2$,

(3) $y \log z = \log 1.5$ nearly, (4) $x \log z = \log 2$.

Dividing (3) by (4) we obtain

$$\frac{y}{x} = \frac{176091}{301030} \text{ nearly,}$$

which, by the method of continued fractions, gives the approximate values

$$\frac{y}{x} = \frac{3}{5}, \frac{7}{12}, \frac{24}{41}, \frac{31}{53}, \frac{179}{306}, \text{\&c.}$$

The approximation $\frac{3}{5}$ gives $y = 3$, $x = 5$; whence by (2) $z = 1.14871$; and by (1) the fifth, $z^y = 1.557$. This error of .0157 is too great to be tolerated by the musical ear. The approximation $\frac{7}{12}$ would give $y = 7$, $x = 12$, $z = 1.05946$, and $z^y = 1.49830$. This is the usual chromatic scale, and as the error in the "fifth" is but .0017 it satisfies the ordinary ear. The approximation $\frac{24}{41}$ would give as the ratio of the fifth $z^y = 1.50042$, an error of only .00042. The approximation $\frac{31}{53}$ would, like $\frac{7}{12}$, give a flatted fifth; its ratio would be $z^y = 1{,}49994$, and this error of .00006 would probably be quite imperceptible even to the nicest ear. An instrument to play 53 notes in an octave would, however, probably be difficult of construction; nor can we expect voices to move with certainty through such small intervals. — M. H. DOOLITTLE, Sophomore Class, Antioch College, Ohio.

Additional Note by Rev. THOMAS HILL, President of Antioch College. — Mr. DOOLITTLE has calculated the values of the approximation $\frac{3}{5}$ as follows: —

1.0000, 1.1487, 1.3195, 1.5157, 1.7411, 2.0000.

The nearest notes to these, in the twelve-semitone scale, are C, D, F, G, B-flat, C, which will instantly be recognized as the scale of B-flat, with the seventh and fourth omitted, that is, the scale of the old Scotch melodies. Thus it appears that this old-fashioned scale approximates rudely towards a division of the octave into five equal divisions. The simple fractions which approximate most nearly to these values are $\frac{8}{7}$, $\frac{4}{3}$, $\frac{3}{2}$, $\frac{7}{4}$, and perhaps the Scotch singer may follow these simpler divisions when singing without a keyed or fretted instrument. If, with Mr. POOLE, we admit the prime seventh into our musical theories, this is a point perhaps worthy of investigation.

Mr. DOOLITTLE's 5-note scale differs from the Scotch in taking C instead of B-flat as the tonic. If we add to it the note A, retaining C as the tonic, it becomes identical with the Irish scale, in which, for example, "Huggamur pene on Sambhruliun" is written.

THE MOTIONS OF FLUIDS AND SOLIDS RELATIVE TO THE EARTH'S SURFACE.

[Continued from Page 97.]

SECTION VI.

ON THE MOTIONS OF THE OCEAN.

72. BESIDES the actions of the sun and moon which give rise to the tides, there are only two causes which can produce any sensible motions on the waters of the ocean. One of these is the action of the atmosphere upon the surface of the ocean, and the other, the difference of density between the water near the equator and that towards the poles, arising from a difference of temperature. The general motions of the atmosphere at the surface of the ocean have a tendency to cause a westward motion of the water in the torrid zone, and an eastward motion in the middle and higher latitudes; and from what we know of the effects of strong winds upon the ocean, we have reason to think that these general motions of the atmosphere are adequate to produce *sensible* motions, since, after the inertia of the water is once overcome, which, however small the force, is only a question of time, the only force necessary is that which is adequate to overcome the resistance of friction, which is very small where the velocity is small. The difference of density between the equator and the poles causes a slight interchanging motion of the water between them, and consequently, where not interrupted by continents, it produces a system of motions in the ocean similar to those of the atmosphere. Hence these two causes

of oceanic disturbance, whatever their relative weight, both act in the same directions, and conjointly cause the observed westward motion of the ocean near the equator, and eastward motion towards the poles.

73. The westward motion of the water of the ocean in the torrid zone was first observed by Columbus, and is now well established; and observations also show that there is a motion towards the east in the higher latitudes. A bottle thrown into the ocean near Cape Horn was picked up three and a half years afterward at port Philip, Australia, a distance of 9000 miles, which makes the eastward velocity in that latitude more than 7 miles per day. And Sir James Ross, when sailing eastward near Prince Edward's Island, found himself every day from 12 to 16 miles by observation in advance of his reckoning. (*Voyage to the Antarctic Seas*, Vol. II. p. 96.) But a westward motion being established in the torrid zone, an eastward motion in the higher latitudes must be admitted; for, as was shown in the case of the atmosphere (§ 35), the one cannot exist without the other.

74. It has generally been supposed that the equatorial westward current of the ocean is caused principally by the action of the westward winds there; but Professor Guyot thinks that "it is too deep and rapid to admit of being explained by their action alone," and that "the difference of temperature between the regions near the equator and those near the poles controls all other causes by its power and the constancy of its action." (*Earth and Man*, pp. 189, 190.) The torsive or deflecting force which causes the westward motion of the atmosphere and the ocean in the equatorial regions, and the eastward motion in the higher latitudes, has been shown to be as the velocity of the interchanging motion between the equatorial and the polar regions; and hence if this motion in both were similar, the relative amount of this force in each must be as the

whole mass multiplied into the velocity of this motion between the equator and the poles. If we suppose the ocean to be 3 miles in depth, its mass is about 500 times that of the atmosphere, and hence if the motion between the equator and the poles were only $\frac{1}{500}$ of that of the atmosphere, the part of the force which gives it a westward motion near the equator, and an eastward motion toward the poles, arising from this cause, must be greater than that of the action of the atmosphere upon it, since the whole amount of this force in the atmosphere is not spent upon the ocean, but only that part which overcomes the resistances to its motions. Although the effect of temperature in producing a difference of density, and consequently of disturbing the equilibrium, is very much less in the ocean than in the atmosphere, yet since the amount of motion which a given disturbing force will produce where time is not considered, depends, as has been stated, upon the amount of the resistances, and not upon the amount of inertia to be overcome; and since the resistances diminish as the square of the velocity, a very small amount of disturbing force arising from a difference of density must be adequate to cause an interchanging motion in the ocean between the equatorial and the polar regions equal to $\frac{1}{500}$ of that of the atmosphere; and hence we have reason to think that a greater part of the motions of the ocean is due to this cause than to the action of the atmosphere upon it.

75. The motions of the ocean being similar to those of the atmosphere, they must cause a slight elevation of the surface about the parallels of 30°, and a depression at the equator and the poles, just as in the case of the atmosphere, except that it will be less in the ratio of the relative velocities of the motions of the ocean and of the atmosphere. If we suppose the east and west motions of the ocean to be $\frac{1}{60}$ of those of the atmosphere at the earth's surface, as given in the third column of the computed table (§ 48), which would

require the maximum eastward velocity in the southern hemisphere to be about 10 miles per day, it would cause the surface of the ocean in the southern hemisphere to be about 15 feet higher at the parallel of 30° than at the pole, and also a little higher than at the equator. Now if the motions which cause this accumulation of water were the same at the bottom of the ocean as at the surface, there would be no tendency of the water to flow out at the bottom from beneath this accumulation; but since the motions there must be much less, it must flow out both toward the equator and the pole, especially toward the latter, as the depression there is much the greater. Since the density of sea-water does not increase below the temperature of 28°, the density of the ocean does not increase beyond a certain latitude, and hence there is no flow of the water at the bottom from the poles toward the equator, arising from the maximum density at the pole, as seems to be the case in a very slight degree in the atmosphere, but the under current at the bottom, arising from the greater pressure about the parallel of 30°, must extend entirely to the poles; so that there must be a slight tendency of the water to rise at the poles, and flow at the surface some distance towards the middle latitudes. As the water toward the bottom of the ocean is always about the same as the mean temperature of the earth, when it first rises to the surface at the pole, it must be much warmer than it is after it has flowed some distance from it, and hence we have reason to think that there may be open polar seas, surrounded by barriers of ice at some distance from the pole, where there is the maximum temperature of the surface water. A surface current from the poles is indicated by the motions of icebergs in both hemispheres from the polar regions towards a lower latitude.

76. Where the east and west motions of the ocean are entirely intercepted by continents, as in the northern hemisphere, the water receives a slight gyratory motion from left to right. The westward

motion of the waters of the Atlantic in the torrid zone, impinging against the continent of America, causes the surface of the water of the Caribbean Sea and the Gulf of Mexico to be a little above the general level, while the eastward motion of the northern part of the Atlantic causes the surface of the water adjacent to the eastern coast of North America, in that latitude, to be a little lower. Hence there is a flow of warm water from the Gulf of Mexico along the coast of the United States toward the lower level about Newfoundland, which, on account of the peculiar configuration of the coast about the Gulf of Mexico and the peninsula of Florida, gives rise to the Gulf Stream. The eastward motion also of the northern part of the Atlantic causes the surface of the water on the western coast of Europe to be a little *higher* than the general level, while the westward motion in the torrid zone causes it to be depressed, on the western coast of Africa, a little *below* this level, and hence the water of the eastern side of the Atlantic, flowing from a higher to a lower level, has a motion toward the equator. The whole of the North Atlantic has therefore a very slight gyratory motion from left to right, and is supposed to make a complete gyration in about three years.

77. A portion of the equatorial current flowing from the higher level of the Caribbean Sea toward Cape Horn causes the Brazil current, which is deflected eastward by the general eastward motion of the Southern Ocean. The east side of the South Atlantic, as well as that of the North Atlantic, seems to have a motion toward the equator. Says Sir James Ross, "There is a current from the Cape of Good Hope along the west coast of Africa 60 miles wide, 200 fathoms deep, with a velocity of one mile per hour, of the mean temperature of the ocean." (*Voyage to the Southern Seas*, Vol. II. p. 35.) This cannot be a portion of the Mozambique current from the warm waters of the Indian Ocean, passing around the Cape of Good

Hope, and giving rise to the equatorial current of the Atlantic, as has been supposed, but must come from the colder waters of the Southern Ocean. Hence the South Atlantic also has a tendency to assume a gyratory motion, and the equatorial current of the Atlantic is merely the equatorial portion of these two gyrations, with perhaps a small part of the Mozambique current passing around the Cape.

78. The general eastward motion of the water of the northern part of the Atlantic, and the consequent depression of the water next the coast of North America, is the cause of the cold current of water flowing from Baffin's Bay and the east coast of Greenland, between the Gulf Stream and the coast of the United States, called the Greenland current. Since the warm water of the Gulf Stream, in flowing northward, is deflected toward the east (§ 32), and that of the Greenland current, in flowing south, tends toward the west, there is no intermingling of the waters of the two currents, but they are kept entirely separate as if divided by a wall, as has been established by the Coast Survey.

79. There must be a motion of the waters somewhat similar to the Gulf Stream and the Greenland current, wherever the great equatorial current impinges against a continent, and the eastward motion toward the poles is interrupted. Hence, on the eastern coast of South America there is the warm Brazil current which has been mentioned, and on the eastern coast of Asia there is the warm China current, flowing toward the north, similar to the Gulf Stream, and the cold Asiatic current insinuating itself between it and the coast, like the Greenland current. On the east coast of Africa, also, there is the Mozambique current flowing south like the Brazil current, and it is also now well established that, east of the Cape of Good Hope, the general tendency of the water is toward the south. This water must mingle with the general eastward current of the South Sea, and hence there is a slight tendency to a gyratory motion in the Indian Ocean also.

80. On the western sides of the continents there is a motion somewhat the reverse of this, and instead of a warm current flowing north, there is a cold one flowing toward the equator, as has been shown to be the case in the Atlantic. Hence, on the west coast of North America there is a flow of colder water along the coast from the north, and on the west coast of South America is Humboldt's current, much colder than the rest of the ocean in the same latitude, both tending toward the equator to join the great westward current there across the Pacific, and to fill up, as it were, the vacuum which this current has a tendency to leave about the equator, on the west coast of America.

81. When a portion of fluid on the earth's surface gyrates from left to right, the deflecting force arising from the earth's rotation being in this case toward the interior, the surface assumes a slightly convex form. If, however, the velocity of gyration were equal to twice that of the earth's rotation multiplied by the cosine of the polar distance, the centrifugal force arising from the gyration would be exactly equal to the centripetal force arising from the earth's rotation, and consequently they would neutralize each other, and if the velocity of gyration were still greater, the surface would be convex, as has been shown in § 30. The water of the North Atlantic having a very small gyratory velocity in comparison with that of the earth's rotation, the interior is a little elevated above the general level, and consequently the pressure upon the bottom increased. If we suppose a circular portion of it, 3,000 miles in diameter, with its centre on the parallel of 30°, to perform a gyration from left to right in three years, equation (50) would give an elevation of five feet in the middle above the level of the external part. This equation, however, on account of the term which has been neglected in the analysis (§ 25), is not strictly applicable to so large a portion of fluid, but still it gives the order of the effect produced. Now the

gyrations which cause this elevation in the middle being principally towards the top, the increased pressure upon the bottom causes the fluid there to flow out on all sides, with a very small velocity, towards the circumference, and hence the water at the surface has a slight tendency to flow in from all sides towards the interior to supply its place. This completely accounts for that vast accumulation of drift and sea-weed, covering a large portion of the interior of the North Atlantic, called the Sargasso Sea. From what has been stated, the North Pacific must also have a slight gyratory motion from left to right, and hence it likewise has its Sargasso Sea.

ON THE MATHEMATICAL THEORY OF HEAT IN EQUILIBRIUM.

By Simon Newcomb, Nautical Almanac Office, Cambridge, Mass.

SECTION I.

TEMPERATURE.

1. It is proposed to present the basis of this portion of the theory of heat in a simple form. We start from the following hypotheses which may be considered as inductions from observation.

First. Every material surface radiates heat at a rate dependent on the nature of its substance, and the amount of heat or caloric which it contains; and the law of direction of this radiation is such that the amount of heat which falls on any point is proportional to the solid angle subtended by the surface at that point.

Second. Every material surface absorbs a certain portion of the heat which radiates on it from other surfaces, transmits another portion, and reflects the remainder.

The term, material surface, is here used in nearly the same sense

as in mechanics, and is understood to mean a layer of matter so thin that whatever heat falls on one side of it will be equally absorbed by, or instantly diffused through, its whole thickness. A body may be considered as composed of an indefinite series of such surfaces, not in actual contact, and through which heat is conducted by successive radiation from particle to particle, or surface to surface. Since in our present paper we consider only heat in equilibrium, we shall not touch upon the laws of conduction, or molecular radiation, but shall suppose that the amount of heat which the external surface of a body radiates into the interior is just equal to that which it receives from it. In this case the temperature of the body is the same as that of its surface.

2. Two surfaces are said to be at the same temperature, when, being brought into contiguity, the amount of heat radiated from each is equal to that absorbed. One of these surfaces may be considered as that of a thermometer, and the temperature of the other will be measured by the temperature of the thermometer when the two are in equilibrium.

Let us now find the condition that two bodies shall be at the same temperature. For this purpose let φh represent the amount of heat radiated from a unit of surface of the first body in a unit of time, and $\varphi' h'$ that radiated from the second. h is supposed to represent the amount of heat contained in a unit of quantity of the body, and φ is a function dependent on the nature of the body. Also, let b and b' represent the fractions of the heat which the surfaces respectively absorb of the whole amount of heat which falls on them. When the heat is in equilibrium between the surfaces, it is evident that the quantity passing from the first surface to the second, by reflection and radiation, must be equal to that passing in the other direction. Represent this quantity by g. Then $b g$ will be the quantity absorbed by the first body in a unit of time, $b' g$ that

absorbed by the second. But the quantity absorbed being equal to that radiated, we have the two equations,

$$\varphi' h' = b' g, \qquad \varphi h = b g;$$

from which we obtain

$$(1) \qquad \frac{\varphi h}{b} = \frac{\varphi' h'}{b'},$$

and these quotients, or any function of them, we may take as the measure of the temperature of the bodies or surfaces in question. I say any function of them, because we can have no absolute measure of temperature, properly speaking. For example, what shall we take as the mean temperature between the freezing and boiling points of water, which we represent by 50° Centigrade, or 122° Fahrenheit? Suppose that we suffer water at the freezing point to absorb just half the heat necessary to raise it to the boiling point, and call its temperature 50° C. Suppose also that we take other substances at 0°, and suffer them to absorb half the heat necessary to raise them to 100°. If now it were found that all substances would under these circumstances exhibit the same temperature by exposure to the thermometer we should be justified in calling their temperature 50°. But such would be far from being the case; and mathematically considered, it is arbitrary what substance we shall select, the temperature of which should furnish our standard of 50°. But it is found that under the circumstances supposed, all *gases* would exhibit sensibly the same temperature, whether the experiment was performed on gas under constant pressure, or under constant volume, and moreover, in the former case the expansion would have been sensibly half of the whole expansion. It also seems highly probable that if we could subject solid substances to constant volume, we should find them to come into the same category with gases. Since, however, in the present paper, we consider only

equal temperatures, no difficulty will arise from the adoption of any arbitrary scale of temperature, we shall therefore take $\frac{\varphi h}{b}$ as the expression for the temperature of any body.

SECTION II.

SPECIFIC HEAT.

3. By the specific heat of a body is understood the amount of heat necessary to raise the temperature of a unit of its mass 1°. More properly it is the value of $\frac{dh}{d\tau}$, τ representing the temperature, and the mass of the body being supposed unity. The specific heat of a body is therefore, putting $\tau = \frac{\varphi h}{b}$,

$$\frac{b\,dh}{d\,\varphi h} = \varrho,$$

or *it is the expression of a relationship between the absorbent power of a body, and the differential of its radiant power.* It will be observed that we have supposed the absorbent power of the body to be constant, as its temperature rises; this is admissible because we cannot distinguish between the heat which a body reflects, and that which it radiates, so that even if the supposition of constant absorbent power be not true, all the phenomena will still be represented by suitably changing the form of the function φ.

To make the above definition of specific heat clear, suppose that $\frac{d\,\varphi h}{dh}$ is exceedingly small, in other words, that supplying the body with a large amount of heat will increase very slightly the amount of heat it radiates. Then the thermometer by being brought into contiguity with the body will need very little additional heat to make its increase of radiation compensate that of the body, and thus be brought into equilibrium with it, and thus a very large accession of heat to the body will make a very slight increase in its tempera-

ture. But it is evident that in this case the above expression for ϱ would be very great.

2d. Suppose now that b is exceedingly small, or that the surface reflects very nearly all the heat which falls on it, and that an increase of heat produces but an average increase in the rapidity with which heat is radiated by the surface. Then, if while the thermometer and the surface of the body are contiguous and in equilibrium, we add a small quantity of heat to the latter, the rapidity with which it radiates heat will be *slightly* increased. But, since nearly all the heat which is either radiated or reflected from the thermometer to the surface of the body is instantly reflected from that surface back to the thermometer, the latter may require a considerable addition of heat to make the additional heat absorbed by the surface equal to the additional amount radiated, and thus the addition of a small absolute amount of heat to the surface would have greatly increased its apparent temperature. But it is evident that in the case supposed the value of ϱ would be very small.

4. The fact, that if we expose a body to an external temperature slightly below its own temperature, the latter will fall just as fast as it would rise if the external temperature were elevated by the same amount above it, is sometimes adduced as indicating a relationship between the radiant and absorbent powers of surfaces in general. But from what has been said it is quite evident that no such relationship exists; moreover, the observed fact (as above expressed) is a necessary result of the hypotheses respecting the radiation and absorption of heat cited in the commencement of the present paper. For, let $\varphi' h'$ represent the radiant power, and b' the absorbent power of the external medium; g the rapidity with which heat passes from the body to the medium, g' the rapidity with which it passes in the opposite direction. We then have

$$g = \varphi h + (1 - b) g'; \qquad g' = \varphi' h' + (1 - b') g.$$

Also for the rapidity with which the body loses heat, we have

$$\frac{dh}{dt} = \varphi h - b g' = {}^{bb'}\!\left(\frac{\varphi h}{b} - \frac{\varphi' h'}{b'}\right) = \frac{b b' (\tau - \tau')}{b + b' - b b'}.$$

If, then, the scale of temperature which we have adopted coincides with that of the experiment, the above-mentioned phenomenon will hold true for all differences of temperature; in any case it will be true when the differences are small.

SECTION III.

TEMPERATURE OF BODIES EXPOSED TO THE SUN.

5. Hitherto we have considered only the phenomena of heat in equilibrium between two contiguous surfaces. Let us now investigate the laws of temperature of bodies exposed only to the heat radiated from a distant centre, as the sun. Our first problem will be to find the temperature of a plane surface exposed perpendicularly to the rays of the sun, and backed by a perfect non-conducting surface, or by another surface in equilibrium with it. If we represent by r the intensity of the radiant heat of the sun, and by b_1 the coefficient of absorption of the surface for perpendicular rays, the condition that the surface shall radiate as much heat as it absorbs gives

$$\varphi h = b_1 r,$$

and for the temperature we have

$$\frac{\varphi h}{b} = \frac{b_1}{b} r.$$

A distinction is made between b_1 and b oecause a surface does not necessarily absorb an equal portion of heat falling on it at every angle. To find the relationship between b_1 and b, suppose a small extent of surface to be placed inside of a closed surface. If the apparent hemisphere formed by the portion of the closed surface seen from one side of the enclosed surface be divided into very nar-

row zones parallel to the latter, it follows from the first hypothesis that the amount of heat radiated on a spherical point on the enclosed surface will be proportional to the apparent area of the zone, and therefore to $\cos\theta$, when θ represents the angle which a line drawn from any point of a zone to the surface makes with the latter. But the amount of heat which falls on any assigned portion of the surface will be proportional to that which would fall on a spherical point multiplied by $\sin\theta$. Wherefore the quantity of heat which will fall on the surface, making angles of incidence between θ and $\theta + d\theta$, will be represented by $\varphi h \sin 2\theta d\theta$. If, then, b_θ represents the absorbent power of a surface for heat radiating on it at an angle θ, its average coefficient of absorption, or b, will be

$$\int_0^{\frac{1}{2}\pi} b_\theta \sin 2\theta\, d\theta = b.$$

If, now, b_θ is the same function of θ for all substances, and if the coefficient of absorption for the rays of the sun is the same as that for rays radiated by a thermometer, or bears the same ratio to it; then the surfaces of all substances ought, under the circumstances supposed, to attain the same temperature. It seems highly probable that this would be the case in nature.

The above expression for the number of rays which a surface, or any small portion of it, receives at any angle of incidence may be regarded as general. Suppose that a surface receives rays from a single source, and takes successively every possible angular position, in other words, that a normal to the surface points successively and equally in every direction of the celestial sphere. It is evident that the *time* spent by this normal at an angular distance of θ' from the radiating point will be proportional to $\sin\theta'$, and that the rapidity with which the surface receives rays, while the normal is at this angle, is proportional to $\cos\theta'$. Whence the number of rays which

it receives will, as before, be proportional to sin $2\theta'$, and sin 2θ, since we have $\theta' = 90° - \theta$.

6. Let there be a solid body, of any form whatever, but not concave in any part, and opaque to the rays of heat, exposed in the planetary spaces to the rays of the sun, supposed parallel, every side of the body being turned rapidly and indiscriminately toward the sun. To find its temperature, we observe that any element of its surface will, on the whole, receive one fourth as much heat from the sun as if it were exposed directly to its rays at right angles. The average quantity of heat radiated on a unit of the surface during a unit of time will then be $\frac{1}{4} r$. The average amount absorbed will be $\frac{1}{4} b r$. The amount radiated, φh. Whence, as the condition of equilibrium, we have

$$\frac{\varphi h}{b} = \tau = \tfrac{1}{4} r.$$

And this we may regard as the average, or normal temperature due to the intensity of the rays of the sun.

7. We now come to a more important question. Suppose that the solid of the preceding paragraph is covered with a very thin layer, partially diathermanous to the rays of the sun, but entirely opaque to those emanating from the body. Let b' represent the coefficient of absorption of the layer with respect to the sun, b'' its coefficient with respect to the heat radiated from the enclosed body; b the coefficient of absorption of the body, supposed the same for all rays. Also, let r' represent the average amount of heat radiated directly from the sun, through the layer into the body in a unit of time, on a unit of surface of the body, φh and $\varphi' h'$ the radiant powers of the body and layer respectively. If, then, we represent, as before, by g the quantity of heat passing each way between the interior surface of the layer, and the exterior surface of the body, we have, in the case of equilibrium,

$$g = \varphi' h' + (1 - b'')\, g + r', \qquad g = (1 - b)\, g + \varphi h;$$

from which we obtain

$$\frac{\varphi h}{b} = \frac{\varphi' h' + r'}{b''}.$$

The condition of equilibrium between the heat radiated from the layer and that received from the sun gives

$$\varphi' h' = \frac{b' r}{4} + r'.$$

We have therefore for the temperature of the body

$$\tau = \frac{\varphi h}{b} = \frac{b' r + 8 r'}{4 b''}.$$

We conclude from this, that by covering a planet with an atmosphere, or other medium, having the property of glass, and probably of our own atmosphere, in being more diathermanous to the radiant heat of the sun than to that of the planet, and also having a very small absorbing power, the temperature of the planet might be raised to any limit. A similar remark will apply to a thermometer placed within a glass case, which ought to rise much higher, on exposure to the sun, than it would if placed in a similar case of rock-salt or wood. If, however, the atmosphere or the case have *not* the above-mentioned property, no possible combination of other properties can make it cause the planet to be above the normal temperature due to the radiant heat of the sun and stars. It is therefore a mistake to suppose that increasing the absorbent power of an atmosphere will increase its heating properties; on the contrary, it is evident from the last equation that such increase would *diminish* its power of raising the temperature of the planet above the normal. It would, however, tend to equalize the temperature at different times during the same day.

ANSWER TO PROF. F. W. BARDWELL'S NOTE "ON THE HORIZONTAL THRUST OF EMBANKMENTS." Vol. II., P. 52.

By CAPT. D. P. WOODBURY, U. S. Corps of Engineers.

PROFESSOR BARDWELL has not I think fully considered the conditions of the problem in question. Equation (1), Vol. I., p. 176, is, I believe, "the formula usually given." It is given by PONCELET and by many others, and has often been extended to a similar problem, the sliding thrust of arches.

Place a book against a vertical wall and apply the horizontal force necessary to keep it there. What sustains the book? Nothing but the friction due to this horizontal force. Incline the wall, the weight of the book will introduce a new element of friction, but we are not at liberty to neglect the first. The whole effect of friction in retarding motion, is equal to the whole normal pressure multiplied by a constant.

The investigation in question was confined to a particular case, though a case of very frequent occurrence, not elsewhere, so far as I know, specially treated. I mean the case in which the surface of the embankment is parallel to the natural slope of earth.

Professor BARDWELL had in view, probably, "the angle of friction." If that were the object sought, his formula would be correct; for then, by the conditions of the problem, the horizontal force would be zero.

If we take into consideration the cohesion of particles, c per unit of surface along the base of the prism tending to slide, equations (1), (2), (4), Vol. 1, p. 176, will be changed as follows: —

$$(1) \qquad F' \sin v + (F' \cos v + Q \sin v) f + \frac{c\,h \sin a}{\sin (a - v)} = Q \cos v.$$

$$(2) \qquad F' = Q \tan (a - v) - \frac{c\,h \sin^2 a}{\sin (a - v) \cos (a - v)}.$$

$$(4) \qquad F' = \tfrac{1}{2} h^2 \sin^2 a - \tfrac{1}{2} h \sin a \left(h \cos a \tan v' + \frac{2c \sin a}{\sin v' \cos v'}\right).$$

The maximum of F' will correspond to the least value of the subtracted quantity, that is to

$$\tan v' = \tan(a - v) = \sqrt{\frac{2c}{hf + 2c}};$$

and the maximum itself is

$$F = \tfrac{1}{2} h^2 \sin^2 a - h \sin a \cos a \sqrt{\frac{2ch}{f} + \frac{4c^2}{f^2}}$$
$$= \tfrac{1}{2} (dp)^2 - \frac{h\sqrt{(2cfh + 4c^2)}}{1 + f^2}.$$

Mathematical Monthly Notices.

A Treatise on Attractions, LAPLACE'S *Functions, and the Figure of the Earth.* By JOHN H. PRATT, M. A., Archdeacon of Calcutta, late Fellow of Gonville and Caius College, Cambridge, and author of "The Mathematical Principles of Mechanical Philosophy." pp. 126. Cambridge: MacMillan & Co., and 23 Henrietta Street, Covent Garden, London. 1860.

This little volume is really a monograph upon the Figure of the Earth, the subjects of Attractions and LAPLACE'S Functions being prefixed for the sake of giving, in the same volume, just what the student will need in the discussion. Short Treatises, on special subjects, like the one before us, and MR. GODFRAY'S on the Lunar Theory already briefly noticed in the Monthly, are becoming quite common in England, and the plan is a good one; for those who wish to study a particular subject can now, in many cases, find it in a small volume of moderate price. Besides, when a subject is treated by itself, in a single volume, the author is more likely to give it symmetry and completeness; and the attention of the student is not diverted by matter which does not belong to the subject. The first chapter gives the attraction of spherical and spheroidal bodies, and shows for what laws of attraction the matter may be considered as condensed into the centre, together with IVORY'S theorem for finding the attraction of an ellipsoid upon an external particle. The second chapter treats of LAPLACE'S Coefficients and Functions, giving the proof that

$$D_f^2 V + D_g^2 V + D_h^2 V = 0, \qquad \text{or } -4\varrho'\pi,$$

according as the attracted particle is or is not a part of the attracting mass, V being the *Potential.* This equation is shown to be true when R, the reciprocal of the distance of the attracted particle from any point in the body, is put in the place of V. The equation in R is then transformed into polar co-ordinates, and the method of expanding R into a series involving LAPLACE'S coefficients is given; and the remainder of the chapter is devoted to

the investigation of the properties of these coefficients. In chapter third LAPLACE's coefficients are used to determine the attraction of bodies nearly spherical; and it is shown that the part of the potential V which pertains to the excess of the attracting mass over a sphere can be expressed in terms of these coefficients. The fourth chapter is devoted to the attraction of bodies neither spherical nor spheroidal, nor nearly so; and it is shown how to calculate the effects of high table lands, and irregular mountain masses, on the plumb-line or spirit-level. The next chapter, which is the first on the Figure of the Earth, investigates this figure considered as a fluid mass, and therefore consisting of nearly spherical strata; and shows that the surface of a homogeneous mass of fluid, in the form of a spheroid, revolving about an axis with uniform velocity, is in equilibrium. In this chapter we also find a review of MR. HOPKINS's argument to show that the crust of the earth is at least one thousand miles thick; with some special investigations confirming this result. The second chapter investigates the figure on the sole hypothesis of the surface being one of equilibrium and nearly spherical; and is essentially the same as that of Professor STOKE's Memoir Published in the *Cambridge Philosophical Transactions* for 1849. The third and last chapter shows how to determine the figure by geodetic operations. In the former chapter the ellipticity was found to be $\frac{1}{300}$ nearly, upon *a priori* grounds; and in the next and last chapter this result is tested by measurement, "by inquiring whether an ellipse can be found with its axis coinciding with that of the Earth, and cutting the plumb-line at stations along it at right angles; and whether the ellipticity of that ellipse is $\frac{1}{300}$." We commend this volume to those who wish to study the subject discussed in it.

Supplementary Researches in the Higher Algebra, by JAMES COCKLE, M. A., F. R. A. S. Read by Rev. ROBERT HARLEY, F. R. A. S., before the Literary and Philosophical Society of Manchester, England, Nov. 29th, 1859. (Abstract communicated to the Mathematical Monthly.)

"In these Supplementary Researches the author extends the elementary formulæ given in § 2 of his original memoir; compares the cyclical and the epimetric views of the function U; and, following the former, is led to a new cyclical theorem which affords an easy demonstration of a proposition asserted in § 28. Mr. COCKLE then applies Mr. HARLEY's cyclical process to the deduction of certain relations between unsymmetric functions; relations attained with a facility which the labor Mr. COCKLE formerly expended upon epimetrics well enables him to appreciate. The author next considers his symbol θ as a rational and symmetric, but otherwise arbitrary, function of four other functions, one of the latter functions, again, being a rational, but otherwise arbitrary, function of four arbitrary symbols, and the remaining three functions being derived from it by the three phases of an interchange which, provided it be of the fourth degree, is otherwise arbitrary. He then expresses the results of all the binary interchanges that can be performed on θ in terms of the single ones (and it should be noticed that from these the results of the ternary and higher interchanges may be obtained), and infers that θ may be regarded as the root of a sextic of which the coefficients are symmetric functions of the four arbitrary symbols. Mr. COCKLE then shows that if we group the six forms of θ two and two, the two members of each group being derivable one from the other by the conjugate interchanges, then the members of a group are inseparable by any interchanges whatever that can be performed upon the arbitrary symbols which enter into θ. So that symmetric functions of symmetric groups may be formed which are unsymmetric in θ, but yet unchanged by any permutations of the four arbitrary symbols. Consequently, if we apply the four arbitrary symbols as multipliers to four of the roots of a quintic, add the products to the fifth root and make the sum a constituent of θ, the symmetric group-function will be a rational function of the fifth root, and therefore the root of a quintic into the coefficients of which the arbitrary

symbols enter symmetrically. In order to give the greatest simplicity to the sextic in θ, the arbitrary symbols may have any suitable values assigned to them, and if we strive after a SYMMETRIC PRODUCT we find that those values are unreal fifth roots of unity. Mr. COCKLE adds, that the method of Symmetric Products has no special affinity for any particular theory of equations, and that although the evanescence of the resolvent product brings it into relation with that of LAGRANGE and VANDERMONDE, yet that better results may be deduced by applying it to EULER's and BEZOUT's theory, and without supposing that product to vanish."

A Treatise on the Calculus of Finite Differences. By GEORGE BOOLE, D. C. L., Honorary Member of the Cambridge Philosophical Society, Professor of Mathematics in the Queen's University, Ireland. Cambridge: MacMillan & Co., and 23 Henrietta Street, Covent Garden, London. 1860. pp. 248.

This work is composed on precisely the same plan as the author's *Treatise on Differential Equations*, a notice of which may be found in a previous number of the Monthly. In the work before us, particular attention is paid to the connection of its methods with those of the Differential Calculus; and it possesses all those peculiar merits as a text-book which were found to characterize the Treatise on Differential Equations; namely, a natural and logical arrangement of the parts of the subject, each part being followed with a careful selection of appropriate examples, the answers to which are collected at the end of the volume. In the brief space at our disposal, we cannot do better than give the heads under which the subject is treated. Chapter I. Nature of the Calculus of Finite Differences; II. Direct Theorems of Finite Differences; III. Of Interpolation; IV. Finite Integration; V. Convergency and Divergency of Series; VI. The Approximate Summation of Series; VII. Equations of Differences; VIII. Equations of Differences of the first Order, but not of the first Degree; IX. Linear Equations with variable Coefficients; X. Of Equations of Partial and of Mixed Differences, and of Simultaneous Equations of Differences; XI. Of the Calculus of Functions; XII. Geometrical Applications.

This summary must suffice to give a general idea of the work, which we especially recommend to those teachers and students who wish to have at least one good work upon each of the various departments of mathematics.

Astronomical Notices, No. 20. Albany, June 30, 1860. — Besides a very complete and valuable article on the Solar Eclipse of July 18, 1860, by R. T. PAINE, Esq., of Boston, we find a letter from PROF. G. P. BOND, from which we extract elements of the Comet discovered at Harvard College Observatory, June 21, by MR. H. P. TUTTLE.

Elements of Comet III., 1860.

H. P. TUTTLE.	T. H. SAFFORD.
$T = 1860$, June 15.76914, Gr. m. t.	$T = 1860$, June 15.4618, Wash. m. t.
$\log q = 9.46238$	$\log q = 9.45862$
$\left.\begin{array}{l}\pi = 160^\circ\ 34'\ 53''\\ \Omega = \ \ 84\ \ 48\ \ 15\\ i = \ \ 79\ \ 19\ \ \ \ 5\end{array}\right\}$ Ap. Eq. Motion direct.	$\left.\begin{array}{l}\pi = 160^\circ\ 31'\ 35''\\ \Omega = \ \ 85\ \ 10\ \ 31\\ i = \ \ 79\ \ 20\ \ 41\end{array}\right\}$ Ap. Eq.

These orbits are computed from Cambridge observations of June 21, 24, 25.

Editorial Items.

We have received the following solutions of the Prize problems in the April number of the Monthly: —

Harriet S. Hazeltine, Worcester, Mass., Probs. I., II.
R. B. Canfield, Columbia College, N. Y., Probs. I., III., IV.
Isaac H. Turrell, Mt. Carmel, Ind., Probs. III., IV., V.
Henry B. Waterman, Yale College, Ct., Probs. I., II., III., IV., V.
E. O. Gibson, Sheshequim, Pa., Probs. I., II., III.
D. G. Bingham, Ellicotteville, N. Y., Probs. III., IV.
William Minto, University of Michigan, Probs. III., IV.
John R. Emery, College of New Jersey, Probs. III., IV., V.
Cadet Arthur H. Dutton, Military Academy, West Point, Probs. I., II., III., IV., V.
Charles Fish, Patten, Me., I., II., III., IV.
George B. Hicks, Cleveland, Ohio, Probs. III., IV., V.
Gustavus Frankenstein, Springfield, Ohio, Probs. III., IV.
John A. Winebrener, Princeton College, N. J., Probs. III., IV., V.
William Hinchcliffe, Barre Plains, Mass., Probs. I., III., IV.
Horace C. Sylvester, Boston, Mass., Prob. III.
F. E. Tower, Amherst College, Mass., Probs. III., IV.
Levi S. Packard, Chatham, N. Y., Probs. I., II., III.
S. J. Baldwin, Chester, N. J., Probs. I., II.
Lewis Foote, Cazenovia Seminary, N. Y., Probs. I., II.
William Simpson, White Plains, N. Y., Probs. I., III., IV.
W. F. Osborne, Wesleyan University, Ct., Probs. III., IV., V.
M. K. Bosworth, Marietta College, Ohio, Probs. III., IV., V.
Hiram L. Gear, Marietta College, Ohio, Probs. III., IV., V.
G. S. Morison, Harvard College, Prob. I.
Asher B. Evans, Madison University, N. Y., Probs. I., II., III., IV., V.
George A. Osborne, Jr., Lawrence Scientific School, Cambridge, Mass., Probs. III., IV., V.
George C. Round, Wesleyan University, Probs. III., IV., V.
W. W. Wetmore, Hamilton College, N. Y., Probs. III., IV., V.
David Trowbridge, Perry City, N. Y., Probs. I., II., III., IV., V.
H. C. Corey, Exeter, N. H., Prob. III.
Stockwell Bettes, Boston, Mass., Probs. III., IV., V.

Harvard Mathematical Prizes. — Two Prizes, of two hundred and fifty dollars each, offered by the Hon. John C. Gray, "to the two members of the Class of 1860 who shall be found, after a special and thorough examination in the Second Term of their Senior year, to have made the greatest proficiency in the study of Pure Mathematics," have been awarded to C. M. Woodward, of Fitchburg, Mass., and C. A. Phillips, of Salem, Mass., by a committee consisting of Mr. J. B. Henck, Dr. B. A. Gould, Mr. J. D. Runkle, and Mr. Chauncey Wright, in connection with the Instructors in the Mathematical department of the College. Mr. Gray has offered the same Prizes, with the same conditions, to members of the Class of 1861.

Boyden Premium. — U. A. BOYDEN, Esq , of Boston, Mass., offers a Premium of one thousand dollars "to any resident of North America, who shall determine by experiment whether all rays of light and other physical rays, are, or are not transmitted with the same velocity." Competitors must transmit their memoirs to WILLIAM HAMILTON, Actuary of the Franklin Institute, Philadelphia, before the first of January, 1862.

Prof. Peirce's Portrait. — It gives us great pleasure to present to our readers in this number of the Monthly an admirable portrait of PROF. BENJAMIN PEIRCE, of Harvard University, engraved by the eminently successful and distinguished artist, H. WRIGHT SMITH, Esq., of Boston, from an excellent daguerrotype taken by Messrs. SOUTHWORTH and HAWES; and to be able to assure all his personal and scientific friends, both at home and abroad, that it is recognized by his family and intimate friends as a most accurate likeness. The daguerreotype was taken just previous to his sailing for Europe, and we can see in the portrait a faint trace of the ill health which it is hoped his six months' absence abroad will entirely remove.

PROF. WILLIAM FERREL and MR. SIMON NEWCOMB, of the Nautical Almanac Office, have been detailed by the Superintendent, Commander C. H. DAVIS, to observe the Solar Eclipse of July 18th, at Cumberland House, a station of the Hon. Hudson's Bay Company on Saskatchewan River.

EDWARD SAWYER, Esq., of Boston, sends us the following Errata in SHORTREDE'S Logarithmic Tables. On page 7, log 3262, for 3.2134840 read 3.5134840; on page 31, log 24451, for .3882996 read .3882966. On page 285, line 2, Mathematical Monthly, Vol. II., for "*acute angles*," read adjacent angles; on page 286, line 9, for $\frac{1}{\sin^2 \omega}$ read $\frac{1}{\sin^2 \omega'}$.

BOOKS RECEIVED.—*Illustrated Catalogue of Philosophical Apparatus.* EDWARD S. RITCHIE, No. 313 Washington Street, Boston. This new edition (1860) of 84 pages, on tinted paper, well printed, and elegant in all respects, contains letters from twenty-six distinguished physicists in different parts of the country who are using MR. RITCHIE'S Apparatus with entire satisfaction. These letters bear ample testimony of mechanical skill, and MR. RITCHIE'S recent election to a Fellowship in the American Academy of Arts and Sciences attests the scientific ability and acquirements with which this skill is directed. *A Treatise on the Calculus of Finite Differences.* — By GEORGE BOOLE, D. C. L., Honorary Member of the Cambridge Philosophical Society, Professor of Mathematics in the Queen's University, Ireland. Cambridge: MacMillan & Co., and 23 Henrietta Street, Covent Garden, London. 1860. pp. 248. *A Treatise on Plane Co-ordinate Geometry* as applied to the Straight-Line and the Conic Sections. With numerous Examples. By I. TODHUNTER, M. A., Fellow and Assistant Tutor of St. John's College, Cambridge. Second edition revised. Cambridge: MacMillan & Co. 1858. pp. 316. *Elements of English Composition.* — Grammatical, Rhetorical, Logical, and Practical. Prepared for Academies and Schools, by JAMES R. BOYD, A. M. New York: A. S. Barnes and Burr, 51 and 53 John Street. 1860. *Class-Book of Botany.* — Being Outlines of the Structure, Physiology, and Classification of Plants. With a Flora of all parts of the United States and Canada. By ALPHONSO WOOD, A. M. New York: A. S. Barnes and Burr. 1860. *Manual of Geology:* — Designed for the use of Colleges and Academies. By EBENEZER EMMONS, State Geologist of North Carolina. Illustrated with numerous engravings. Second edition. New York: A. S. Barnes and Burr. 1860. *Elements of Analytical Geometry and the Differential and Integral Calculus.* — By CHARLES DAVIES, LL. D., Professor of Higher Mathematics, Columbia College, N. Y. 8vo. pp. 398. New York: A. S. Barnes and Burr. 1860.

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